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A SIMPLIFIED ANALYSIS OF THE EFFECT OF TRANSVERSE SHEAR ON THE RESPONSE OF ELASTIC PLATES TO IMPACT LOADING

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Abstract—This investigation deals with the problem of transverse impact on an elastic plate for which the effect of deformation due to transverse shear cannot be neglected, e.g. a thick plate or one reinforced with aligned fibres in its plane. Closed-form solutions have been obtained, after minor approximation, to determine the deflection and bending moment at the point of impact. These solutions are independent of boundary conditions in the "early state" of the impact phenomenon. Further, using Hertz's contact law, it is possible to determine the history of the impact force produced when a compact body strikes the plate. It is shown that a single parameter can describe the influence of transverse shear on the impact force as well as on the deflection and bending moment at the impact point. Finally, numerical results are presented to show that the influence of shear on deflection is much smaller than that on impact force or bending moment.

NOTATION

D	flexural rigidity of plate, $Eh^3/12(1-\nu^2)$
E, E_s	Young's moduli of plate and the impacting body, respectively
F	impact force
f	dimensionless impact force
G_t	shear modulus of plate in transverse direction
h	thickness of plate
K	Hertz's constant
M_x, M_y	components of bending moment per unit length
M_{xy}	twisting moment per unit length
M_0	bending moment at the point of impact
M_0^*	dimensionless bending moment at the point of impact
M_s	mass of the impacting body
q	distributed load per unit area on the surface of the plate
r_s	radius of curvature of the impacting body near the point of impact
T_0	reference time
t, t'	time, impact occurs at $t = 0$
v_0	velocity of the impacting body at $t = 0$
w	deflection of the mid-plane of the plate
w_0	deflection at the point of impact
w_0^*	dimensionless deflection at the point of impact
w_s	displacement of the impacting body
x, y	Cartesian coordinate axes, located in the mid-plane of the plate. Point of impact is (0, 0)
α, β	components of rotation of the mid-plane
α_p	plate parameter, $(1/8)\sqrt{(1/Dh\rho)}$
β'	shear parameter
λ	impact parameter
ν, ν_s	Poisson's ratios of plate and the impacting body, respectively
ρ	mass density of the plate material
τ, τ'	dimensionless time.

1. INTRODUCTION

Most of the previous studies[1–8] dealing with the response of elastic plates subjected to impact loads are based on the classical plate equation in which the effects of transverse shear and rotatory inertia are neglected. In these studies series solutions or closed-form solutions based on some simplifying assumptions have been obtained to determine the deflection and bending strain when a plate is subjected to a time-dependent load. Moreover, using the deflection so obtained and Hertz's contact law it is possible to predict the contact force developed when an elastic body impacts on the plate. Schwieger[6] and Mittal *et al.*[8] have experimentally verified the predictions of the classical theory for the case of a thin aluminium plate struck by a steel ball at its centre. It was reported that while there is a good agreement between experimental and predicted values of the maximum contact force and maximum deflection, the agreement for maximum bending strain was not satisfactory.

A recent investigation on beams[9] has shown that transverse shear, which depends upon the radius of gyration of the beam cross-section and the ratio E/G_t of Young's modulus in the longitudinal direction and shear modulus in the thickness direction, modifies the dynamic response of the beam. Since the ratio E/G_t can be quite high for fibre-reinforced plastics (FRP) and drop-weight tests are often recommended for the dynamic characterization of FRP structural components, the need for the present investigation was felt.

The aim of the present work is to obtain closed-form solutions, without neglecting shear effects, for the deflection and bending moment caused in a plate subjected to a time-dependent load. Further, on the basis of these results, it is proposed to obtain the history of the contact force between the plate and the striking mass. Lastly, this effect of transverse shear on the response of a plate is clearly brought out by considering two practical examples.

2. BASIC EQUATIONS AND ANALYSIS

We consider a plate which is transversely isotropic in the x–y plane, i.e. $E_x = E_y = E$; $\nu_{xy} = \nu_{yx} = \nu$; $G_{xy} = E/2(1+\nu)$ and $G_{xz} = G_{yz} = G_t$. The flexural motion of this plate, including the effect of shear deformation is governed by the following differential equations[10]:

$$\alpha + \frac{\partial w}{\partial x} - \frac{6D}{5G_t h}\left[\frac{\partial^2 \alpha}{\partial x^2} + \frac{1-\nu}{2}\frac{\partial^2 \alpha}{\partial y^2} + \frac{1+\nu}{2}\frac{\partial^2 \beta}{\partial x \partial y}\right] = 0 \tag{1}$$

$$\beta + \frac{\partial w}{\partial y} - \frac{6D}{5G_t h}\left[\frac{\partial^2 \beta}{\partial y^2} + \frac{1-\nu}{2}\frac{\partial^2 \beta}{\partial x^2} + \frac{1+\nu}{2}\frac{\partial^2 \alpha}{\partial x \partial y}\right] = 0 \tag{2}$$

$$\frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y} + \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} - \frac{6\rho}{5G_t}\frac{\partial^2 w}{\partial t^2} + \frac{6q}{5G_t h} = 0. \tag{3}$$

Mindlin[11] has shown that the classical theory of plates with shear corrections is sufficiently close to the exact three-dimensional theory. The addition of rotatory inertia terms increases the mathematical complexity but produces only a very small improvement. For shear correction Mindlin used a factor of $\pi^2/12$ instead of 5/6 in eqns (1)–(3). Some authors[12] used 2/3.

The bending and twisting moments, per unit length are given by

$$M_x = D\left(\frac{\partial \alpha}{\partial x} + \nu\frac{\partial \beta}{\partial y}\right); \qquad M_y = D\left(\frac{\partial \beta}{\partial y} + \nu\frac{\partial \alpha}{\partial x}\right)$$
$$M_{xy} = \frac{G_{xy}h^3}{12}\left(\frac{\partial \alpha}{\partial y} + \frac{\partial \beta}{\partial x}\right). \tag{4}$$

For classical impact problems Eringen[4] introduced a kernel (Green's function) so that the deflection is obtained as a linear homogeneous functional of the impact force history. The kernel itself is obtained as a series solution of the corresponding free vibration problem. Further, several authors[5–8] have approximated the series to obtain a closed-form solution for the plate problem when the shear effects are neglected. This approximation is valid in the "early state" of the impact phenomenon, i.e. till the fastest flexural waves return, after reflection at the boundaries, to the point under consideration. It has been analytically shown, as may be intuitively expected, that the exact nature of the boundary conditions has no influence on the dynamic behaviour of the plate during the "early state". In particular, if the impact phenomenon lasts for a period much less than the time taken for the return of the fastest flexural waves to the point of impact then the plate may be assumed to be infinite in extent. This statement has been theoretically verified[9] for a beam subjected to a concentrated load at its centre irrespective of whether the shear effects are considered or not.

Therefore, in the present work the plate has been assumed to be infinite and a concentrated time-dependent load $F(t)$ is applied at $x = 0$, $y = 0$. Thus in eqn (3)

$$q(x,y,t) = F(t)\,\delta(x-0)\,\delta(y-0) \tag{5}$$

where δ represents the Dirac-delta function. For this plate the solution of eqns (1)–(3) can be easily obtained using the Fourier transform method[13]. Introduce the following Fourier transforms:

$$A(\xi,\eta,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \alpha(x,y,t)\,e^{i(\xi x+\eta y)}\,dx\,dy$$

$$B(\xi,\eta,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \beta(x,y,t)\,e^{i(\xi x+\eta y)}\,dx\,dy$$

$$W(\xi,\eta,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w(x,y,t)\,e^{i(\xi x+\eta y)}\,dx\,dy \tag{6}$$

and

$$Q(\xi,\eta,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} q(x,y,t)\,e^{i(\xi x+\eta y)}\,dx\,dy$$
$$= \frac{F(t)}{2}$$

where eqn (5) has been used for simplifying the last transform. Using the plate equations, eqns (1)–(3) it is easily shown that

$$A = \frac{-i\xi W}{1+br^2}; \qquad B = \frac{-i\eta W}{1+br^2} \tag{7}$$

and

$$W(\xi,\eta,t) = \frac{1}{2\pi\rho h}\int_0^t F(t')\frac{\sin\omega(t-t')}{\omega}\,dt' \tag{8}$$

where

$$b = \frac{6}{5}\frac{D}{G_t h}; \qquad r^2 = \xi^2+\eta^2 \tag{9}$$

and

$$\omega = \sqrt{\left(\frac{D}{\rho h}\right)}\left\{\frac{r^2}{\sqrt{(1+br^2)}}\right\}. \tag{10}$$

The deflection $w(x,y,t)$ can be obtained by taking the reverse transform of eqn (8). Thus

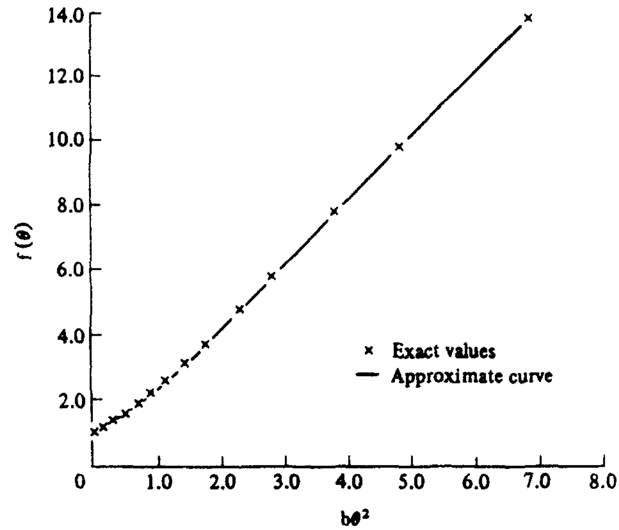

Fig. 1. Comparison of the exact values of $f(\theta)$ with the approximate curve representing $f(\theta) = e^{-b\theta^2} + 2b\theta^2$.

$$w(x, y, t) = \frac{1}{(2\pi)^2 h\rho} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(\xi x + \eta y)} \int_0^t F(t') \frac{\sin \omega(t-t')}{\omega} \, dt'. \tag{11}$$

In particular the deflection at the point of loading is

$$w_0(t) = w(0, 0, t) = \frac{1}{(2\pi)^2 h\rho} \int_0^t F(t') \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\sin \omega(t-t')}{\omega} \, d\xi \, d\eta \tag{12}$$

where it has been assumed that the conditions for the change of order of integration hold. Further, the double integral appearing in eqn (8), denoted by I, is rewritten in terms of polar coordinates (r, Φ). After integration with respect to Φ, we get

$$I = 2\pi \int_0^{\infty} \frac{\sin \omega(t-t')}{\omega} r \, dr. \tag{13}$$

This integral cannot be evaluated directly and, therefore, we seek an approximating function for the integrand in the same manner as discussed in Ref. [9]. The procedure consists of replacing r by a new variable θ such that

$$\theta^2 = \frac{r^2}{\sqrt{(1 + br^2)}} \tag{14}$$

and then rewriting the integrand as an explicit function of θ. This necessitates expressing $2(1+br^2)^{3/2}/(2+br^2)$ as a function $f(\theta)$ of θ only. A close approximation has been found numerically for $f(\theta)$ which is valid for $0 \leqslant \theta < \infty$. Thus

$$f(\theta) \simeq e^{-b\theta^2} + 2b\theta^2. \tag{15}$$

This approximation is very close, approaching the exact values when $\theta \to 0$ as well as when $\theta \to \infty$. A discernible difference occurs only in a small interval near $b\theta^2 = 2.0$. The exact and approximate values of $f(\theta)$ for $0 \leqslant \theta \leqslant 8$ are shown in Fig. 1. In view of the above-mentioned replacements and introducing the notation $a = \sqrt{(D/\rho h)}$ and $T = t - t'$ we have

$$I \simeq \frac{2\pi}{a}\int_0^\infty \frac{1}{\theta}\sin\,(aT\theta^2)\,\{e^{-b\theta^2}+2b\theta^2\}\,d\theta. \tag{16}$$

The integration of the first term in eqn (16) is easily obtained while the second term presents some difficulties. Its value fluctuates between 0 and $2/aT$. The same integral has been investigated in Ref. [8] where it was shown that a weak convergence for this integral can be obtained and the converged value is $1/aT$. A similar result can be obtained from a physically more intuitive viewpoint. Suzuki[14] has considered the impact of a rigid mass on a viscoelastic circular plate for which he has shown that the amplitude of the central deflection decays exponentially and the rate of decay is higher for higher modes. Thus, the converged value of the second integral of eqn (16) can be obtained by considering the limiting operation

$$\lim_{\Phi\to 0}\int_0^\infty e^{-\Phi y}\sin\,(aTy)\,dy$$

which again equals $1/aT$. Substituting the value of I in eqn (12) we finally get

$$w_0 = \frac{1}{2\pi h\rho a}\int_0^t F(t')\left\{\frac{1}{2}\tan^{-1}\left[\frac{a(t-t')}{b}\right]+\frac{b}{a(t-t')}\right\}dt'. \tag{17}$$

Equation (17) yields the classical result when shear effects are neglected, i.e. $b = 0$. Then it is easily seen that

$$w_0 = \frac{1}{8}\sqrt{\left(\frac{1}{Dh\rho}\right)}\int_0^t F(t')\,dt'. \tag{18}$$

The term $(1/8)\sqrt{(1/Dh\rho)}$ is known as the plate parameter (α_p). This result was first obtained by Boussinesq[1] using a different method.

Now we consider the expressions for the bending moment. Taking Fourier transforms on both sides of eqn (4) and using expressions (6) we get after some simplification

$$\bar{M}_x = \frac{WD}{1+b(\xi^2+\eta^2)}\{\xi^2+\nu\eta^2\}$$

and (19)

$$\bar{M}_y = \frac{WD}{1+b(\xi^2+\eta^2)}\{\eta^2+\nu\xi^2\}$$

where $\bar{M}_x$ and $\bar{M}_y$ are the Fourier transforms of M_x and M_y, respectively. For an isotropic or a transversely isotropic plate $M_x = M_y$ at the point of loading. This is always true for an infinite plate but true only in the "early state" for a finite plate. Due to this equality of bending moments it can be shown that $\bar{M}_x = \bar{M}_y = \frac{1}{2}(\bar{M}_x+\bar{M}_y)$. Then the Fourier transform $\bar{M}_0$ of M_0 is given as

$$\bar{M}_0 = \tfrac{1}{2}(\bar{M}_x+\bar{M}_y) = \frac{WDr^2}{1+br^2}\left(\frac{1+\nu}{2}\right) \tag{20}$$

where $r^2 = \xi^2+\eta^2$ as before. Hence at the point of loading

$$M_0(t) = \frac{(1+\nu)}{2}\frac{D}{2\pi}\int_{-\infty}^{t}\int_{-\infty}^{t}\frac{Wr^2}{1+br^2}\,d\xi\,d\eta \tag{21}$$

$$= \frac{(1+\nu)D}{4\pi\rho h}\int_0^t F(t')\int_0^\infty \frac{\sin\omega(t-t')}{\omega}\frac{r^3}{1+br^2}\,dr.$$

Now consider the integral

$$J = \int_0^\infty \frac{\sin\omega(t-t')}{\omega}\frac{r^3}{1+br^2}\,dr \tag{22}$$

$$= \frac{1}{a}\int_0^\infty \sin\left(\frac{aTr^2}{\sqrt{(1+br^2)}}\right)\frac{r\,dr}{\sqrt{(1+br^2)}}$$

where a and T have been defined earlier. This integral is rewritten (without any approximation) as

$$J = \frac{1}{2a}\int_0^\infty \sin(aT\theta)\left[1+\frac{b\theta}{\sqrt{(4+b^2\theta^2)}}\right]d\theta$$

where θ has been defined in eqn (14). Using integral tables (p. 473, rule 5 of Ref. [15]) and after substitution and simplification we finally get

$$M_0(t) = \frac{1+\nu}{8\pi}\left[\int_0^t \frac{F(t')}{t-t'}\,dt' + \frac{2a}{b}\int_0^t F(t')K_1\left\{\frac{2}{b}a(t-t')\right\}dt'\right] \tag{23}$$

where K_1 is the modified Bessel's function of the second kind and first order. Again when shear effects are neglected, i.e $b = 0$ the second term vanishes since $K_1(\infty) = 0$. In that case we regain the classical result, namely

$$M_0(t) = \frac{1+\nu}{8\pi}\int_0^t \frac{F(t')}{t-t'}\,dt'. \tag{24}$$

Equations (17), (23) and (24) involve integrals of the type

$$\int_0^t \frac{1}{t-t'}\,dt'$$

which is not convergent. Schwieger[6] has overcome this difficulty by replacing the upper limit of the integral t by $t-t_0$ where t_0 is a positive time constant, which was taken as $(h^2/16)\sqrt{(\rho/D)}$. This expression was arrived at by comparing the results of Schwieger's theory with those obtained by Sneddon[3] for a plate impact problem in which the impact load is distributed over a small area instead of being a point load. The justification for this comparison is that in an actual plate a small region of the plate near the point of application of the load will experience very high contact stresses and possibly some plastic deformation which will spread the load over a non-zero area. Therefore, in the present theory Schwieger's correction will be applied.

3. CALCULATION OF IMPACT FORCE

For the calculation of the impact force produced during impact due to a compact mass, we use Hertz's theory for the contact force between two bodies. Since the time of contact between the striking body (assumed to be spherical for the sake of simplicity) and the plate is usually much larger than the time period of the lowest modes of vibration of the sphere, Hertz's theory is still applicable although it considers only a static contact between the bodies (see Rayleigh[16]). According to Hertz's theory, the relationship between impact force $F(t)$ and the relative approach $(w_s - w_0)$ is given as

$$F(t) = K[w_s(t) - w_0(t)]^{3/2} \tag{25}$$

where for a flat plate

$$K = \frac{4}{3} \frac{E_s E r_s^{1/2}}{E_s(1-\nu^2) + E(1-\nu_s^2)}. \tag{26}$$

Also the equation of motion of the ball is

$$F(t) = -m_s \frac{\partial^2 w_s}{\partial t^2}. \tag{27}$$

The value of $F(t)$ can be obtained from eqns (25), (27) and (17). The resulting integral equation is

$$K^{-2/3}\{F(t)\}^{2/3} = v_0 t - \frac{1}{m_s}\int_0^t F(t')\,(t-t')\,\mathrm{d}t'$$
$$-\frac{1}{2\pi h\rho a}\int_0^t F(t')\left[\tfrac{1}{2}\tan^{-1}\left\{\frac{a(t-t')}{b}\right\} + \frac{b}{a(t-t')}\right]\mathrm{d}t'. \tag{28}$$

This equation can be rewritten in the following non-dimensional form:

$$f^{2/3}(\tau) - \tau + \int_0^\tau f(\tau')\left[(\tau-\tau') + \frac{4\lambda}{\pi}\left\{\tfrac{1}{2}\tan^{-1}\left(\frac{\tau-\tau'}{\beta'}\right) + \frac{\beta'}{\tau-\tau'}\right\}\right]\mathrm{d}\tau' = 0 \tag{29}$$

where $\tau = t/T_0$ and $\tau' = t'/T_0$ are dimensionless times such that the reference time T_0 is given by the following expression:

$$T_0 = \frac{m_s^{2/5}}{K^{2/5} v_0^{1/5}}. \tag{30}$$

Moreover, the dimensionless force $f(\tau)$ is given as

$$f(\tau) = \frac{T_0}{m_s v_0} F(t) = K^{-1}(v_0 T_0)^{-3/2} F(t). \tag{31}$$

Equation (29) also contains two parameters

$$\lambda = \frac{\alpha_p m_s}{T_0} \quad \text{and} \quad \beta' = \frac{b}{a T_0} = \frac{9.6\rho\lambda}{m_s}\frac{D}{G_t}. \tag{32}$$

For an isotropic plate

$$\beta' = \frac{1.6}{1-\nu}\frac{\lambda h^3 \rho}{m_s}$$

where λ is the impact parameter and β' represents shear effects. Interestingly, for an isotropic

Table 1. Variation of f_{max} and τ_c as functions of λ and β' for $\lambda \leqslant 1.0$

β'	$\lambda = 0.2$ f_{max}	τ_c	β'	$\lambda = 0.5$ f_{max}	τ_c	β'	$\lambda = 1.0$ f_{max}	τ_c
2.0	0.476	5.7	0.5	0.51	5.0	0.25	0.423	5.4
1.0	0.614	4.6	0.1	0.687	3.8	0.10	0.490	4.8
0.5	0.74	4.0	0.05	0.719	3.7	0.05	0.524	4.5
0.2	0.856	3.5	0.02	0.737	3.6	0.02	0.543	4.4
0.1	0.901	3.4	0.01	0.743	3.6	0.01	0.549	4.4
0.0	0.945	3.3	0.0	0.75	3.6	0.0	0.556	4.4

Table 2. Variation of f_{max} as a function of λ and β' for $\lambda \geqslant 2.0$

β'	λ 2.0	5.0	10.0	20.0
0.1	0.317	0.154	—	—
0.05	0.339	0.164	0.088	0.046
0.01	0.359	0.174	0.093	0.048
0.005	0.362	0.176	0.094	0.049
0.0	0.365	0.177	0.095	0.049

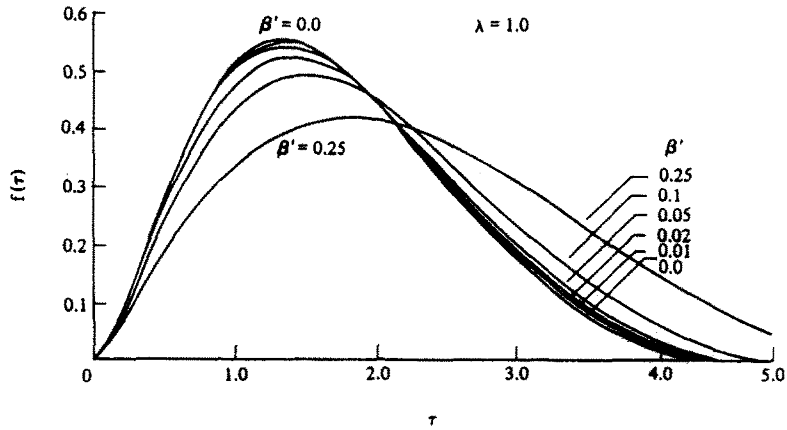

Fig. 2. Variation of dimensionless impact force $f(\tau)$ with dimensionless time τ for $\lambda = 1.0$ and various values of the shear parameter β'.

plate the shear parameter is proportional to the ratio of the mass of a cube of the plate material with side h and the mass of the striking ball.

Equation (29) is a non-linear integral equation which can be solved numerically by the step-by-step method due to Timoshenko[17]. The last term of this equation shows a singularity as $\tau \to \tau'$. Therefore, the appropriate size of steps was chosen by examining the convergence of the solution $f(\tau)$ by progressively reducing their size. It was found that a size of 0.01 yielded $f(\tau)$ quite close to the converged value and at the same time the computation time was not very high. Solutions were obtained for $\lambda = 0.2, 0.5, 1.0, 2.0, 5.0, 10.0$ and 20.0. The choice of β' was not arbitrary but was chosen on the basis of the dimensions of plates and elastic constants of materials used in experiments conducted by many authors. It was observed that for most plates $\beta\lambda < 1.0$. A summary of the important results of numerical computations for all the choices of λ and β' that were considered is given in Tables 1 and 2. Also, Figs 2 and 3 show the variation of dimensionless impact force $f(\tau)$ as a function of dimensionless time τ for two values of λ. Comparing the two figures one observes some similarities and some dissimilarities for the $\lambda = 1.0$ and 10.0 cases. In the former case, β' influences the entire impact force history while in the latter case the influence is limited to a narrow region around the maximum. In both cases the impact force rises to a maximum and then decreases. However, for the $\lambda = 1.0$ case the decrease is much faster while for the $\lambda = 10.0$ case it is very gradual. As a result, the time

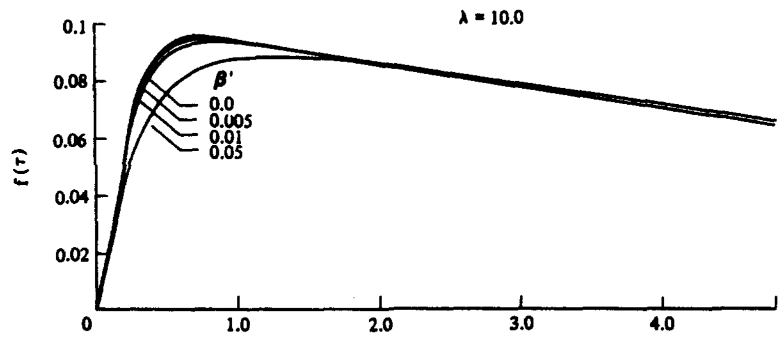

Fig. 3. Variation of $f(\tau)$ with τ for $\lambda = 10.0$.

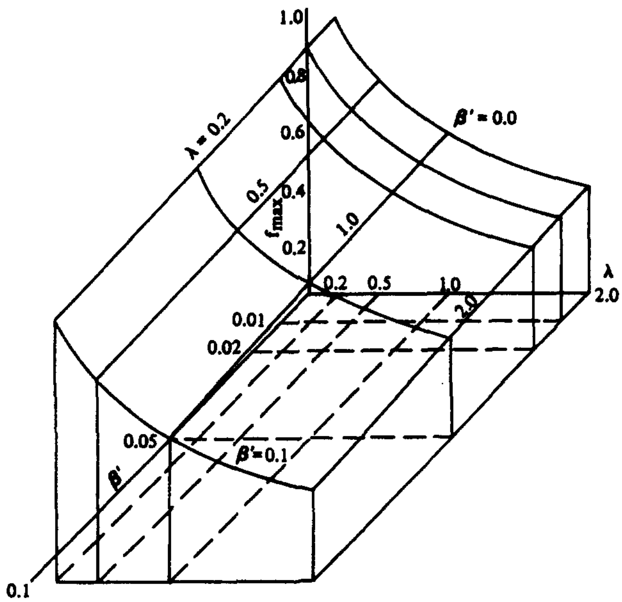

Fig. 4. A three-dimensional plot of the maximum dimensionless impact force f_{max}, impact parameter λ and shear parameter β'.

of contact between the ball and the plate for $\lambda = 1.0$ is much smaller than for $\lambda = 10.0$. In Table 1 the maximum impact force f_{max} and the time of contact τ_c, both in dimensionless forms, are shown as functions of the impact parameter λ and the shear parameter β'. It is seen that both these parameters strongly influence f_{max} and τ_c. For higher values of λ, the dimensionless contact time τ_c exceeds 6.0. When the contact time is large the results of the present theory should be used with caution because the flexural waves may return to the point of impact after reflection at the boundaries of the plate. Therefore, in Table 2, only the values of f_{max} are given for various values of λ and β'.

Since λ and β' are the characteristic parameters of an impact problem it is necessary to have a rational procedure to predict the maximum impact force for any values of λ and β'. Figure 4 shows a three-dimensional plot of f_{max} as a function of λ and β', for $0 < \lambda \leqslant 2.0$ and $0 \leqslant \beta' \leqslant 0.1$. Similar plots can be drawn for any other intervals of λ and β'. It is clearly seen from Fig. 4 that in the range of β' considered here, f_{max} is almost linearly dependent on β' for a fixed λ. The dependence on λ is, however, nonlinear. Using interpolation, it is possible, to obtain f_{max} with sufficient accuracy for any values of λ and β'. Another effect of the presence of shear is that the occurrence of the maximum impact force is progressively delayed with increasing β'. This can be confirmed by referring to Figs 2 and 3.

4. CALCULATION OF DEFLECTION AND BENDING MOMENT AT IMPACT POINT

After obtaining the impact force history the deflection and bending moment at the point of impact can be calculated with the help of eqns (17) and (23), respectively. The influence of

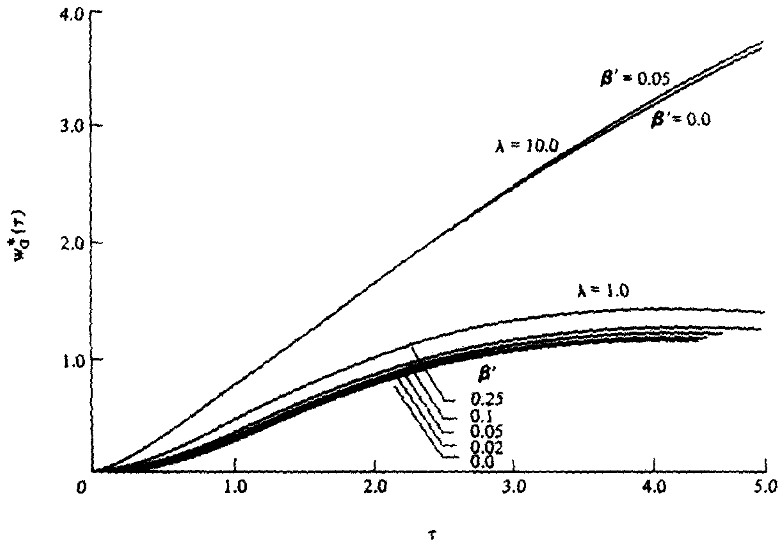

Fig. 5. Variation of dimensionless deflection w_0^* at the impact point with τ for various λ and β'.

β' on dimensionless deflection w_0^* for $\lambda = 1.0$ and 10.0 is shown in Fig. 5. The dimensionless deflection is given as

$$w_0^*(\tau) = \frac{4\lambda}{\pi}\int_0^\tau f(\tau')\left\{\frac{1}{2}\tan^{-1}\left(\frac{\tau-\tau'}{\beta'}\right)+\frac{\beta'}{\tau-\tau'}\right\}\,\mathrm{d}\tau. \tag{33}$$

It is related to the actual deflection w_0 at the impact point by the following relation:

$$w_0(t) = v_0 T_0 w_0^*(\tau). \tag{34}$$

As in the case of impact force, the influence of β' on deflection is negligible in the case when $\lambda = 10.0$ while it is quite discernible when $\lambda = 1.0$.

Next we consider the evaluation of the bending moment at the point of loading. The numerical integration of eqn (23) is facilitated by rewriting this equation in the following dimensionless form:

$$M_0^*(\tau) = -\int_0^\tau \frac{f(\tau')}{\tau-\tau'}\{1+zK_1(z)\}\,\mathrm{d}\tau' \tag{35}$$

where

$$M_0^*(\tau) = \frac{8\pi T_0}{(1+\nu)m_s v_0}M_0(\tau) = \frac{8\pi}{1+\nu}\frac{M_0(\tau)}{K(v_0T_0)^{3/2}} \tag{36}$$

and

$$z = 2(\tau-\tau')/\beta'. \tag{37}$$

The computer evaluation of the modified Bessel's function $K_1(z)$ was carried out by a series expansion as given by Abramowitz and Stegun[18]. The integral of eqn (35) consists of two terms

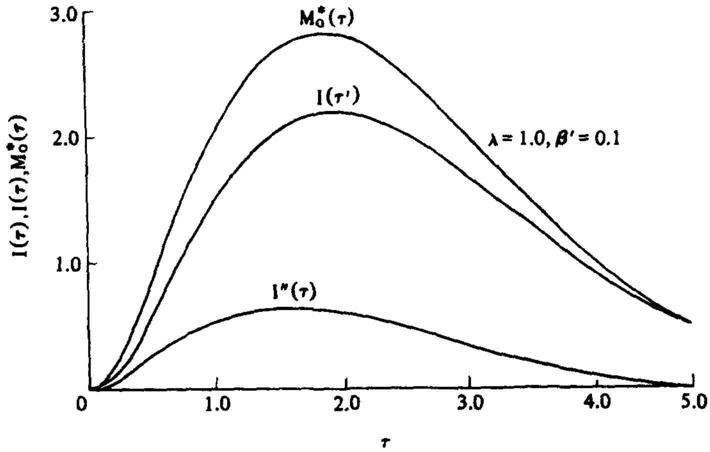

Fig. 6. Variation of dimensionless bending moment M_0^* at the impact point with τ for $\lambda = 1.0$ and $\beta' = 0.1$. I' and I'' represent the contributions of the classical theory and the shear effects, respectively.

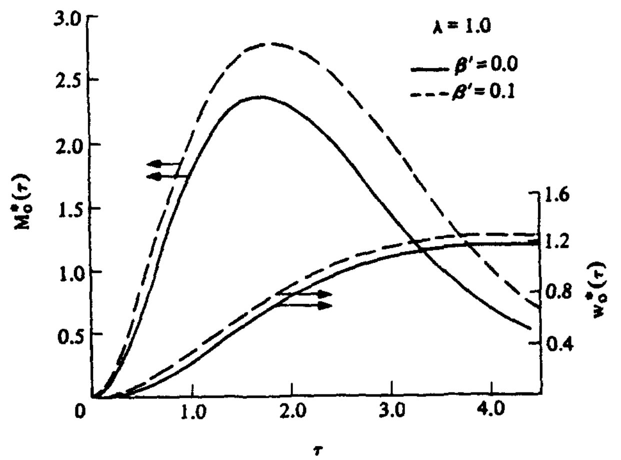

Fig. 7. Comparison of $M_0^*(\tau)$ and $w_0^*(\tau)$ for the classical and present theories when $\lambda = 1.0$.

$$I'(\tau) = \int_0^\tau \frac{f(\tau')}{\tau - \tau'} \, d\tau'$$

and

$$I''(\tau) = \int_0^\tau \frac{F(\tau')}{\tau - \tau'} [zK_1(z)] \, d\tau'.$$

These two integrals were evaluated separately to emphasize the effect of transverse shear on the bending moment M_0^*. The magnitudes of I' and I'' for the case when $\lambda = 1.0$ and $\beta' = 0.1$ are shown in Fig. 6. It is seen that I'' is not always negligible and in this case the maximum contribution of I'' amounts to about 30% of I'. The total dimensionless bending moment $M_0^*(\tau)$ is also shown in the same figure.

Finally, it may be stated that the influence of transverse shear on the load-point bending moment is exerted in two ways. First, the transverse shear influences the impact force history and secondly, for the given impact force history there is one extra term contributing to the bending moment calculations. In the classical theory only integral I' is present.

5. CONCLUSIONS

In order to clearly bring out the main conclusions of the present investigations, Figs 7 and 8 are useful where the deflection and bending moment histories (both in dimensionless

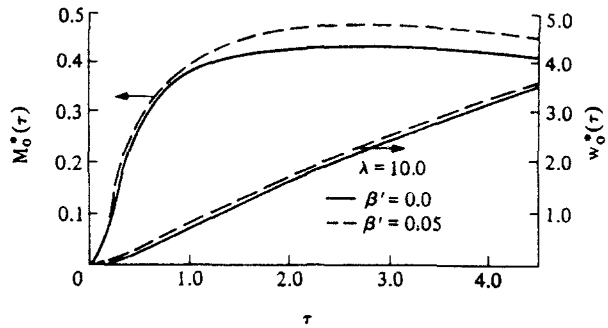

Fig. 8. Comparison of $M_0^*(\tau)$ and $w_0^*(\tau)$ for the classical and present theories when $\lambda = 10.0$.

form) are compared with their classical counterparts. Again, the same two values of λ, i.e. 1.0 and 10.0 were chosen. The shear parameters for the two cases were 0.1 and 0.05, respectively. Classical cases are represented by $\beta' = 0.0$. On the basis of Figs 2, 3, 7 and 8 the following main conclusions are drawn:

(1) β' has a significant influence on the impact force and bending moment and only a marginal influence on the deflection under the point of loading.

(2) Both these effects are higher for a lower value of λ.

(3) The maximum force is higher for the classical case, as compared to the case when shear effects are taken into account. On the other hand the maximum bending moment is lower for the classical case. In fact, the bending moment and deflection are lower at all times when shear effects are neglected. The results of our analysis agree qualitatively with those of Filipov and Skylar[12] who used the series method to investigate the shear and rotatory inertia effects on the impact behaviour of finite rectangular plates.

(4) As mentioned in Section 2, the results of this analysis are valid for a period lasting till the return to the impact point of flexural waves after reflection at the boundaries of the plate.

The conclusions of the present investigations are directly relevant to the analysis of drop-weight tests on thick plates of metals and alloys and even for moderately thick plates of fibre-reinforced composites since the value of β' is high. The experimental data of these tests is generally analysed on the basis of the classical plate theory ($\beta' = 0$) and hence it can lead to erroneous values of material parameters such as dynamic flexural strength, etc.

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